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EMBEDDING UNCONDITIONAL STABLE BANACH SPACES INTO SYMMETRIC STABLE BANACH SPACES

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ABSTRACT

In this paper we prove the following result which solves a question raised by A. Pelczynski: "Every stable Banach space with an unconditional basis is isomorphic to a complemented subspace of some stable Banach space with a symmetric basis." Moreover, we show that all the interpolation spaces $(l^p, l^q)_{\theta,X}$, $1 \leqq p,\ q < \infty$ and X stable, are stable.

J. Lindenstrauss (cf. [6]) showed that every Banach space X with an unconditional basis is isomorphic to a complemented subspace of a space Y with a symmetric basis. A. Szankowski [11] proved that if X is reflexive then Y can be constructed reflexive. W. J. Davis [3] obtained the same type of result when X is uniformly convex. Here, we will show that if X is stable, in the sense of Krivine–Maurey, Y is also stable. (It solves a question raised by A. Pelczynski to the author.) We will use the approach of Davis, which in essence is an interpolation method.

All notations are standard: X denotes a Banach space and $(e_i)_{i=1}^{\infty}$ a basis of X. If $x \in X$ we put $x = \Sigma_1^{\infty} x(i)e_i$. We represent by $\operatorname{supp} x = \{i \in \mathbf{N};\ x(i) \neq 0\}$ and $|A| =$ cardinality of a set A.

If $x, y \in X$, $x \wedge y = 0$ signifies that x and y are disjoint.

If $(E_i)_i$ is a sequence of Banach spaces and X is a Banach space with an unconditional basis (e_i), $\bigoplus_X E_i$ is the space of the E_i-valued sequences $f = (f(i))_i$, with $f(i) \in E_i$ and such that $\Sigma_1^{\infty} \|f(i)\|_{E_i} e_i \in X$. For $f \in \bigoplus_X E_i$ we define $\|f\| = \|\Sigma_1^{\infty} \|f(i)\| e_i\|$ ($\|\cdot\|$ represents different norms in different spaces, but this does not cause any problem).

A separable Banach space X is stable in the sense of Krivine and Maurey [5],

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if

$$\lim_{n\mathcal{U}} \lim_{m\mathcal{V}} \| x_n + y_m \| = \lim_{m\mathcal{V}} \lim_{n\mathcal{U}} \| x_n + y_m \|$$

whenever $(x_n)_n$, $(y_m)_m$ are bounded sequences in X and $\mathcal{U}$, $\mathcal{V}$ are non-trivial ultrafilters on $\mathbf{N}$.

A function σ of X into R^+ is a type on X if there exist a bounded sequence $(x_n)_n$ in X and a non-trivial ultrafilter $\mathcal{U}$ on $\mathbf{N}$, such that

$$\sigma(x) = \lim_{n\mathcal{U}} \| x + x_n \|.$$

We denote by $\tau(X)$ the set of the types on X.

Our goal in this paper is to prove the following theorem:

1. THEOREM. *Given $\varepsilon > 0$, every stable Banach space with a 1-unconditional basis is $(1+\varepsilon)$-isomorphic to a complemented subspace of some stable Banach space with a 1-symmetric basis.*

We use the approach of Davis in the construction of the 1-symmetric Banach space, as it appears in [7]. Let E and F be two Banach sequence spaces so that the unit vectors $(e_k)_k$ form a 1-symmetric basis in both E and F. We assume that the canonical embedding $F \xrightarrow{i} E$ is of norm 1 and

$$\lim_{n\to\infty} \left\| \sum_1^n e_i \right\|_E \Big/ \left\| \sum_1^n e_i \right\|_F = 0.$$

If $m \geqq 1$ we may define a new norm on E

$$\| \alpha \|_m = \inf\{(\| \beta \|_E^2 + \| \gamma \|_F^2)^{1/2};\ \alpha = m^{-1}\beta + m\gamma,\ \beta \in E,\ \gamma \in F\}.$$

This norm is a quotient norm in the following sense. Let $\|\cdot\|'$ be the norm defined in $F \oplus E$ by putting

$$\|(\gamma, \beta)\|' = (m^{-2}\| \gamma \|_F^2 + m^2 \| \beta \|_E^2)^{1/2}, \qquad \gamma \in F, \quad \beta \in E,$$

then, $(E, \|\cdot\|_m)$ is isometrically isomorphic to the quotient space $F \oplus E / V$ where $V = \{(\gamma, -i(\gamma));\ \gamma \in F\}$ (note that V is a closed subspace of $F \oplus E$ because it is the graph of $-i$).

It is not difficult to refine the proof of Proposition 3.b.4 of [7] to get

2. PROPOSITION. *Given $\varepsilon > 0$ and a Banach space X with a normalized 1-unconditional basis, there exists an increasing sequence of numbers $(m_n)_n$ with $\Sigma_1^\infty m_n^{-1} < \infty$ such that the space Y, the diagonal of $\oplus_X (E, \|\cdot\|_{m_n})$, contains a complemented subspace $(1+\varepsilon)$-isomorphic to X.*

We apply the above proposition to $F = l^p$, $1 < p < 2$, and $E = l^2$. We have to prove that, for all m, $(l^2, \|\cdot\|_m)$ is stable, and besides, that $\bigoplus_X E_i$ is stable, whenever E_i and X are stable. It is well known ([2], [8]) that the quotients of l^p are stable if $p > 1$; here, we also prove that all quotients of $l^p \bigoplus l^q$ are stable ($1 < p, q < \infty$). In [1] the stability of $\bigoplus_X E_i$ is shown when X is 1-symmetric Banach space. We will extend this result to "sums" defined by stable Banach spaces with unconditional basis.

In the first place, we are going to study the stability of $\bigoplus_X E_i$. The next lemma is due to B. Maurey and it enables us to improve a preliminary version of Proposition 4 (we had to suppose that X had a finite cotype). We wish to thank B. Maurey for allowing us to reproduce his argument here.

3. Lemma. *Let X be a Banach space with an unconditional basis $(e_i)_i$. Suppose that X does not contain any subspace isomorphic to c_0. Let $(x_n)_n$ be a bounded sequence in X, then, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} = y_k + w_k$, $k = 1, 2, \ldots$, with $y_k \wedge w_k = 0$, $(y_k)_k$ converges in X and $(w_k)_k$ is a block basis sequence.*

Proof. We may suppose $(e_i)_i$ is 1-unconditional and, by passing to a subsequence, $(x_n)_n$ converges coordinatewise to an element $x_\infty \in X$ (because X does not contain c_0). The proof is based on the following claim:

Claim. *If $\varepsilon > 0$, we can find two increasing sequences of integers $(j_k)_0^\infty$ and $(n_k)_1^\infty$ such that, for all k,*

$$\left\| \sum_{i=0}^{j_0} x_{n_k}(i) e_i - \sum_{i=0}^{j_0} x_\infty(i) e_i \right\| < \varepsilon$$

and

$$\left\| \sum_{j_0+1}^{j_k} x_{n_k}(i) e_i \right\| < \varepsilon.$$

By applying the claim for $\varepsilon = 1$, we obtain $\mathbf{N}_1 = (n_k^{(1)})_1^\infty$ and $I_1 = (j_k^{(1)})_0^\infty$; we can apply the claim again to the subsequence $(x_{n_k^{(1)}})_k$ and $\varepsilon = \frac{1}{2}$, and besides, we may choose $j_k^{(2)} > j_k^{(1)}$ for all k; then, we repeat inductively the construction for $\varepsilon = 1/k$.

Let $\{n_1, n_2, \ldots\}$ be the diagonal of $\mathbf{N}_k$'s. Let $\{j_0, j_1, \ldots\}$ be the diagonal of I_k's. By construction $(j_k)_0^\infty$ increases. Now, we take $J_k (> j_k)$ verifying

$$\left\| \sum_{J_k}^{\infty} x_{n_k}(i) e_i \right\| < \frac{1}{k}$$

for all k. By defining

$$y_k = \sum_{i=0}^{j_k} x_{n_k}(i)e_i + \sum_{J_k}^{\infty} x_{n_k}(i)e_i$$

it is clear that $(y_k)_k$ converges, since $(\|\Sigma_0^{j_0^{(k)}} x_{n_k}(i)e_i\|)_k$ is a Cauchy sequence and

$$\left\| \sum_{j_0^{(k)}+1}^{j_k} x_{n_k}(i)e_i \right\| < \frac{1}{k}.$$

Eventually, by passing to another subsequence, we get the result of the lemma.

PROOF OF THE CLAIM. There exists j_0 such that $\|\Sigma_{j_0+1}^{\infty} x_\infty(i)e_i\| < \varepsilon$. The sequence $(\Sigma_1^{j_0} x_n(i)e_i)_n$ converges to $\Sigma_1^{j_0} x_\infty(i)e_i$, then,

$$\left\| \sum_1^{j_0} x_n(i)e_i - \sum_1^{j_0} x_\infty(i)e_i \right\| < \varepsilon \qquad \text{whenever } n \geqq \nu.$$

We denote $j_k = j_0 + k$; since

$$\lim_{n\to\infty} \left\| \sum_{j_0+1}^{j_k} x_n(i)e_i \right\| < \varepsilon,$$

we may inductively choose n_k, with $\nu < n_1 < n_2 < \cdots$ verifying

$$\left\| \sum_{j_0+1}^{j_k} x_{n_k}(i)e_i \right\| < \varepsilon. \qquad \square$$

Let us now prove the stability of $\bigoplus_X E_i$.

4. PROPOSITION. *Let X be a stable Banach space with an unconditional basis $(e_i)_i$ (of course, X does not have a subspace isomorphic to c_0). Let $(E_i)_i$ be a sequence of stable Banach spaces, then $\bigoplus_X E_i$ is stable.*

PROOF. If $f \in \bigoplus_X E_i$ we denote by f the corresponding element $\bar{f} = \Sigma_i \|f(i)\| e_i \in X$. Let $(f_n)_n$, $(g_m)_m$ be two bounded sequences in X, then we have to prove that

$$\lim_{n\mathcal{U}} \lim_{m\mathcal{V}} \|f_n + g_m\| = \lim_{m\mathcal{V}} \lim_{n\mathcal{U}} \|f_n + g_m\|.$$

Since the sequences $(\bar{f}_n)_n$ and $(\bar{g}_m)_m$ are bounded in X by passing to subsequences, we can split each $\bar{f}_n$, $\bar{g}_n$, $\bar{f}_n = \bar{f}_n^{(1)} + \bar{f}_n^{(2)}$, $\bar{g}_m = \bar{g}_m^{(1)} + \bar{g}_m^{(2)}$, $n, m \in \mathbf{N}$ where the sequences $(\bar{f}_n^{(1)})_n$, $(\bar{g}_m^{(1)})_m$ converge in X, $(\bar{f}_n^{(2)})_n$ and $(\bar{g}_m^{(2)})_m$ are block basis sequences, and $\bar{f}_n^{(1)} \wedge \bar{f}_n^{(2)} = 0$, $\bar{g}_m^{(1)} \wedge \bar{g}_m^{(2)} = 0$. We denote by $f_n^{(2)}$, $g_m^{(2)}$ the vectors of $\bigoplus_X E_i$, defined by $\bar{f}_n^{(2)}$ and $\bar{g}_m^{(2)}$.

Given $\varepsilon > 0$, there exist natural numbers ν and M such that

$$\left\| \bar{f}_n^{(1)} - \sum_1^{\nu} \| f_n(i) \| e_i \right\| < \varepsilon,$$

$$\left\| \bar{g}_m^{(1)} - \sum_1^{\nu} \| g_m(i) \| e_i \right\| < \varepsilon,$$

for all $n, m \geqq M$. We define α_n, $\beta_m \in \bigoplus_X E_i$ $(n, m \in \mathbf{N})$ by putting

$$\alpha_n(i) = f_n(i) \qquad \text{if } 1 \leqq i \leqq \nu,$$

$$\beta_m(i) = g_m(i) \qquad \text{if } 1 \leqq i \leqq \nu,$$

$\alpha_n(i) = \beta_m(i) = 0$ if $i > \nu$. Now, we only have to show that

$$\lim_{n\mathcal{U}} \lim_{m\mathcal{V}} \| \alpha_n + f_n^{(2)} + \beta_m + g_m^{(2)} \| = \lim_{m\mathcal{V}} \lim_{n\mathcal{U}} \| \alpha_n + f_n^{(2)} + \beta_m + g_m^{(2)} \|.$$

But if n and m are quite far apart

$$\| \alpha_n + f_n^{(2)} + \beta_m + g_m^{(2)} \| = \| \overline{\alpha_n + \beta_m} + \bar{f}_n^{(2)} + \bar{g}_m^{(2)} \|.$$

Since E_i's are stable and the vectors $\alpha_n + \beta_m$ are finite dimensional, the limits $\lim_{n\mathcal{U}} \lim_{m\mathcal{V}} (\alpha_n + \beta_m)$, $\lim_{m\mathcal{V}} \lim_{n\mathcal{U}} (\alpha_n + \beta_m)$ exist and are equal.

Thus the stability of $\bigoplus_X E_i$ inherits the stability of X. □

The preceding proposition is a new method of constructing stable Banach spaces. Now we study the stability of $l^p \oplus l^q / V$. Because not all the norms on $l^p \oplus l^q$ are stable (see [8], p. 11), we only consider norms given by "norms of $\mathbf{R}^2$", that is, if $\| \cdot \|^*$ is a norm on $\mathbf{R}^2$, we define

$$\| (x, y) \| = \| (\| x \|_p, \| y \|_q) \|^*,$$

$x \in l^p$, $y \in l^q$, which is a norm giving the usual topology on $l^p \oplus l^q$.

In order to show the stability of the quotients of $l^p \oplus l^q$ we use the following result proved by Y. Raynaud. We wish to thank Y. Raynaud for communicating his result to us. This theorem has permitted us to simplify the preliminary version of the proof of Proposition 6.

5. Proposition. [10] *Let X be a reflexive stable Banach space. If the cone formed by the weakly-null types is strongly locally compact, then, for every closed subspace the corresponding quotient space is stable.*

The strong topology in the space of the types is the topology defined by uniform convergence on bounded subsets of X. A type is weakly-null if it can be defined by a weakly convergent to zero sequence.

Let X be a Banach space with an unconditional basis $(e_i)_i$ and suppose that X does not contain any subspace isomorphic to c_0. As a consequence of Lemma 3, we may give an asymptotic representation of the types on X. If σ is a type, then

$$\sigma(x) = \lim_{n\mathcal{U}} \|x + a + a_n\|, \qquad x \in X,$$

where a is a fixed vector of X and $(a_n)_n$ is a block basis sequence. Since $a = \text{w-lim}_n (a + a_n)$, then the weakly-null types have the following simpler form:

$$\sigma(x) = \lim_{n\mathcal{U}} \|x + a_n\|,$$

$x \in X$, where $(a_n)_n$ is a block basis sequence. If the unconditional constant of the basis $(e_i)_i$ is 1, then the class of weakly-null types coincides with the class of symmetric types. Indeed, all weakly-null types are symmetric because, if X has finite support and n is larger than some n_0, x and a_n are disjoint and, then, $\|x + a_n\| = \|-x + a_n\|$; for arbitrary x, the result holds by approximation. Conversely, let σ be a symmetric type. If $\mu = \lim_{n\mathcal{U}} \|a_n\|$, we have

$$\nu = \sigma(-a) = \lim_{n\mathcal{U}} \|2a + a_n\| \qquad \text{(by 1-unconditionality)}$$

$$= \lim_{n\mathcal{U}} \|-2a + a_n\| = \sigma(3a) = \lim_{n} \|4a + a_n\| = \cdots.$$

Hence $\nu = \lim_{n\mathcal{U}} \|2ka + a_n\|$ for all $k \in \mathbf{N}$. Thus $a = 0$. For instance, in l^p, the symmetric types are

$$\sigma(x) = (\|x\|^p + \mu^p)^{1/s}, \qquad x \in X$$

and then they are l^p-types (see [2], [4]).

If we consider $l^p \oplus l^q$ with the norm defined before, a weakly null type has the representation given by

$$\sigma(x, y) = \|([\|x\|_p^p + \mu^p]^{1/p}, [\|y\|_q^q + \nu^q]^{1/q})\|,$$

$x \in l^p$, $y \in l^q$, $\mu, \nu \geqq 0$.

6. Proposition. *Let V be a closed subspace of $l^p \oplus l^q$, $1 < p, q < \infty$. Then the quotient space $l^p \oplus l^q / V$ is stable.*

Proof. Let $\sigma^{(n}$ be a suite of weakly-null types on $l^p \oplus l^q$ such that

$\sigma^{(n}(0,0) \leqq M$, $n \in \mathbf{N}$; we must show that $\sigma^{(n} \to_n \sigma$ uniformly on bounded subsets. If $\sigma^{(n}$ is defined by the two scalars μ_n, ν_n, it is clear that $(\mu_n)_n$ and $(\nu_n)_n$ are bounded sequences on $\mathbf{R}$. Let σ be the type defined by $\mu = \lim_n \mu_n$ and $\nu = \lim_n \nu_n$. Thus

$$|\sigma(x,y) - \sigma^{(n}(x,y)| \leqq C\|(|\mu^p - \mu_n^p|^{1/p}, |\nu^p - \nu_n^p|^{1/p})\| \xrightarrow[n]{} 0$$

uniformly on (x, y). □

An argument, the same as the one given in the proof of Proposition 5, shows that the result is true for stable dual spaces if we consider quotients by w*-closed subspaces and suppose that the cone of w*-null types is strongly locally compact. Hence Proposition 6 can be extended to the quotients of $l^p \oplus l^q$, $1 \leqq p, q < \infty$ by w*-closed subspaces.

The proof of Theorem 1 is now an immediate consequence of the above propositions.

Remarks. (1) If X is an unconditional stable Banach space then we only can assure that X is isomorphic to a complemented subspace of some symmetric stable Banach space.

(2) The proof of Propositions 4 and 6 ensures that all the interpolation spaces $(l^p, l^q)_{\theta,X}$ $(1 \leqq p \leqq q < \infty$, X an unconditional stable Banach space) are stable. Indeed $(l^p, l^q)_{\theta,X}$ is the diagonal of $\bigoplus_{X,m=0}^{\infty} E_m$, where the spaces E_m are l^q with the norms given by the gauge of $e^{\theta m}B_p + e^{-m(1-\theta)}B_q$ (B_p and B_q are the unit balls of l^p and l^q). But the gauge $\|\cdot\|_m$ of $e^{\theta m}B_p + e^{-m(1-\theta)}B_q$ can be represented by means of

$$\|x\|_m = \inf\{\max[e^{-\theta m}\|y\|_p, e^{m(1-\theta)}\|x-y\|_q];\ y \in l^p\}.$$

Thus, $(l^q, \|\cdot\|_m)$ is isometrically isomorphic to the quotient space $l^p \oplus l^q / V$, where $V = \{(y, -i(y));\ y \in l^p\}$, i is the canonical embedding $l^p \to l^q$ and $l^p \oplus l^q$ is endowed with the norm

$$\|(y,x)\| = \max\{^{-\theta m}\|y\|_p, e^{m(1-\theta)}\|x\|_q\}$$

if $y \in l^p$, $x \in l^q$. Consequently $(l^q, \|\cdot\|_m)$ is stable and so is $(l^p, l^q)_{\theta,X}$.

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